NOTES ON HYPERSURFACE GEOMETRY AND RELATED ANALYSIS

This manuscript provides some elementary geometric and analytic details relevant to evolution of domains in Euclidean spaces arising in some problems of incompressible fluids. The two sections are parallel and they may refer to each other. It will be updated from time to time. Suggestions are welcome.

Notations. Most of the notations will be defined as they are introduced. In addition a list of symbols will be given at the end of the paper for a quick reference. Here we'll present some standard notations and conventions used throughout the paper.

All constants will be denoted by C which is a generic bound depending only on the quantities specified in the context. We follow the Einstein convention where we sum upon repeated indices.

For a domain Ω_t and $x \in \partial\Omega_t$ we denote by $N(t,x)$ the outward unit normal, Π the second fundamental form where $\Pi(w) = \nabla_w N \in T_x\partial\Omega_t$ for $w \in T_x\partial\Omega_t$, and κ the mean curvature given by the trace of Π, i.e., $\kappa = \mathrm{tr}\Pi$. The regularity of the domains Ω_t is characterized by the local regularity of $\partial\Omega$ as graphs. In general, an m-dimensional manifold $\mathcal{M} \subset \mathbb{R}^n$ is said to be of class C^k or H^s, $s > \frac{n}{2}$, if, locally in linear frames, $\mathcal{M}$ can be represented by graphs of C^k or H^s mappings, respectively. For $\partial\Omega$, throughout this paper we will only use these local graph coordinates in orthonormal frames.

1. Geometry of evolving domains

Suppose $\Omega_t \subset \mathbb{R}^n$ is a family of smooth domains with the parameter t, moving with a smooth velocity vector field $v(t,x)$, $x \in \Omega_t$. We calculate various quantities related to the evolution of the geometry of the domain, which are essential in the energy estimate of the free boundary problem of the Euler's equations.

1.1. **Material derivative $\mathbf{D}_t$.** For any $x_0 \in \bar{\Omega}_{t_0}$, the particle path $x(t)$ is the solution of the ODE:

$$x_t = v(t,x) \qquad x(t_0) = x_0$$

and the material derivative $\mathbf{D}_t = \partial_t + \nabla_v$ is differentiation along the direction of $x(t)$ in the space time domain in $\mathbb{R}^n \times \mathbb{R}$. Clearly, $x(t) \in \partial\Omega_t$ if $x_0 \in \partial\Omega_{t_0}$.

Calculations of $\mathbf{D}_t N$. At any $x_0 \in \partial\Omega_{t_0}$, $\mathbf{D}_t N(t_0, x_0) \perp N(t_0, x_0)$ since $|N(t,x)| \equiv 1$. To derive $\mathbf{D}_t N(t_0, x_0)$, let $\tau(t) \in T_{x(t)}\partial\Omega_t$ be a solution to the linearized particle path ODE:

$$\mathbf{D}_t \tau = \nabla_\tau v \qquad\qquad \tau(t_0) = \tau_0 \in T_{x_0}\partial\Omega_{t_0}.$$

At (t_0, x_0), $\mathbf{D}_t N \cdot \tau_0 = \mathbf{D}_t(N\cdot\tau) - N \cdot \mathbf{D}_t\tau = -(Dv)^*(N)\cdot\tau_0$. Therefore, we have

$$\mathbf{D}_t N = -((Dv)^*(N))^\top. \tag{1.1}$$

Calculation of $\mathbf{D}_t dS$. For any $x \in \partial\Omega_t$, et $\{y^1, \cdots, y^{n-1}\}$ be a local coordinate systems of a neighborhood of x on $\partial\Omega_t$. Let $G = (g_{ij})_{(n-1)\times(n-1)}$ where $g_{ij} =< \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} >$, then the area element

$$dS = \sqrt{\det(G)}dy^1dy^2\cdots dy^{n-1}.$$

Thus,

$$\mathbf{D}_t dS = \frac{\mathbf{D}_t(\det(G))}{2\sqrt{\det(G)}}dy^1dy^2\cdots dy^{n-1} = \frac{1}{2}\mathrm{tr}((\mathbf{D}_t G)G^{-1})dS.$$

Since the ratio between $\mathbf{D}_t dS$ and dS is coordinate independent, which is the determinant of the linearization of the flow map generated by the vector field v, we may simplify our calculation at x by requiring that $\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^{n-1}}\}$ form an orthonormal basis at x. Using $\mathbf{D}_t \frac{\partial}{\partial y^i} = \nabla_{\frac{\partial}{\partial y^i}} v$, we obtain, at x,

$$\mathbf{D}_t \mathrm{tr}(G) = \mathbf{D}_t g_{ii} = 2 < \nabla_{\frac{\partial}{\partial y^i}} v, \frac{\partial}{\partial y^i} >= 2 < \nabla_{\frac{\partial}{\partial y^i}} v^\top, \frac{\partial}{\partial y^i} > +2 < \nabla_{\frac{\partial}{\partial y^i}}(v^\perp N), \frac{\partial}{\partial y^i} > .$$

It implies

$$\mathbf{D}_t dS = (v^\perp\kappa + \mathcal{D}\cdot v^\top)dS. \tag{1.2}$$

Covariant differentiation $\mathbf{D}_t^\top$. For the family of hypersurfaces $\partial\Omega_t$ with the velocity field v we define parallel transport along the material line $x(t)$ as follows. Given a tangent vector $\tau_0 \in T_{x_0}\partial\Omega_{t_0}$, let $\tau(t)$ be the solution of the following ODE:

$$\mathbf{D}_t\tau \perp T_{x(t)}\partial\Omega_t \Leftrightarrow \mathbf{D}_t\tau = (\nabla_\tau v\cdot N)N, \qquad \tau(t_0) = \tau_0. \tag{1.3}$$

It is easy to verify that $\tau(t) \in T_{x(t)}\partial\Omega_t$ and that this transport preserves the inner product.

A natural connection between $T\partial\Omega_t \subset \mathbb{R}^n$ for different t along the materials lines is provided by the above parallel transport which induces the covariant differentiation $\mathbf{D}_t^\top$, the projection of $\mathbf{D}_t$ in $\mathbb{R}^n$ acting on $w \in T_{x(t)}\partial\Omega_t \subset \mathbb{R}^n$. This covariant differentiation induces the covariant differentiations of linear (multilinear) operators on tensor products of $T\partial\Omega_t$ and $T^*\partial\Omega$, which will also denoted by $\mathbf{D}_t^\top$. For example, given a family of linear operators $L(t)$ on $T_{x(t)}\partial\Omega_t$ and $\tau(t) \in T_{x(t)}\partial\Omega_t$,

$$(\mathbf{D}_t^\top L)(\tau) = \mathbf{D}_t^\top(L(\tau)) - L(\mathbf{D}_t^\top \tau).$$

Calculation of $\mathbf{D}_t^\top \Pi$ and $\mathbf{D}_t\kappa$. Given $\tau \in T_{x_0}\partial\Omega_{t_0}$, let $\tau(t)$ be its parallel transport along the material line $x(t)$ which enable us to compute

$$(\mathbf{D}_t^\top \Pi)(\tau) = \mathbf{D}_t^\top(\Pi(\tau)) = (\mathbf{D}_t \nabla_\tau N)^\top = (\nabla_\tau \mathbf{D}_t N + \nabla_{[\mathbf{D}_t,\tau]} N)^\top = \mathcal{D}_\tau \mathbf{D}_t N + \Pi(([\mathbf{D}_t, \tau])^\top)$$

From (1.3) and (1.1), we have

$$[\mathbf{D}_t, \tau] = \mathbf{D}_t \tau - \nabla_\tau(\frac{\partial}{\partial t} + v) = (\nabla_\tau v \cdot N)N - \nabla_\tau v = -(\nabla_\tau v)^\top, \tag{1.4}$$

$$(\mathbf{D}_t^\top \Pi)(\tau) = -\mathcal{D}_\tau \left(((Dv)^*(N))^\top\right) - \Pi((\nabla_\tau v)^\top). \tag{1.5}$$

To calculate $\mathbf{D}_t\kappa$ at (t_0, x_0), we take an orthonormal frame $\{\tau_1, \ldots, \tau_{n-1}\}$ of $T_{x_0}\partial\Omega_{t_0}$ and parallel transport it into an orthonormal frame along $x(t)$. Thus $\mathbf{D}_t\kappa = \mathbf{D}_t(\Pi(\tau_i)\cdot\tau_i) = (\mathbf{D}_t^\top\Pi)(\tau_i)\cdot\tau_i$ and (1.5) give slightly different but useful forms for $\mathbf{D}_t\kappa$

$$\mathbf{D}_t\kappa = -\mathcal{D}\cdot((Dv)^*(N))^\top - \Pi(\tau_i)\cdot\nabla_{\tau_i} v = -\Delta_{\partial\Omega} v \cdot N - 2\Pi\cdot((D^\top|_{T\partial\Omega_t})v) \tag{1.6}$$

$$\mathbf{D}_t\kappa = -\Delta_{\partial\Omega} v^\perp - v^\perp |\Pi|^2 + (\mathcal{D}\cdot\Pi)(v^\top). \tag{1.7}$$

Calculations of commutators involving $\mathbf{D}_t$. In the following, we will calculate the commutators of $\mathbf{D}_t$ with operators $\mathcal{H}$, $\mathcal{N}$, and $\Delta_{\partial\Omega}$, to show that they are of lower orders.

- $[\mathbf{D}_t, \mathcal{H}]f = \Delta^{-1}(2Dv\cdot D^2 f_{\mathcal{H}} + \nabla f_{\mathcal{H}}\cdot\Delta v)$.

To start, we write the basic formula for any function $f(t,x)$, $x \in \Omega_t$,

$$\mathbf{D}_t \nabla f = \nabla \mathbf{D}_t f - (Dv)^*(\nabla f). \tag{1.8}$$

For the tangential gradient, using $\nabla^\top f = \nabla f - (\nabla_N f)N$, it is straight forward to obtain

$$\mathbf{D}_t^\top \nabla^\top f = \nabla^\top \mathbf{D}_t f - ((Dv)^*(\nabla^\top f))^\top \tag{1.9}$$

Let $f(t,x)$, $x \in \partial\Omega_t$, be a smooth function. Recall $f_{\mathcal{H}} = \mathcal{H}(f)$ represents the harmonic extension of f into Ω_t. We have

$$\Delta \mathbf{D}_t f_{\mathcal{H}} = \mathbf{D}_t \Delta f_{\mathcal{H}} + 2\nabla v \cdot D^2 f_{\mathcal{H}} + \nabla_{\Delta v} f_{\mathcal{H}} = 2Dv \cdot D^2 f_{\mathcal{H}} + \nabla f_{\mathcal{H}} \cdot \Delta v \tag{1.10}$$

which implies

$$\mathbf{D}_t f_{\mathcal{H}} = \mathcal{H}(\mathbf{D}_t f) + \Delta^{-1}\Delta \mathbf{D}_t f_{\mathcal{H}} = \mathcal{H}(\mathbf{D}_t f) + \Delta^{-1}(2Dv \cdot D^2 f_{\mathcal{H}} + \nabla f_{\mathcal{H}} \cdot \Delta v).$$

Therefore we can write

$$\mathbf{D}_t \mathcal{H}(f) = \mathcal{H}(\mathbf{D}_t f) + \Delta^{-1}(2Dv \cdot D^2 f_{\mathcal{H}} + \nabla f_{\mathcal{H}} \cdot \Delta v). \tag{1.11}$$

- $[\mathbf{D}_t, \Delta^{-1}]g = \Delta^{-1}(2Dv \cdot D^2\Delta^{-1}g + \Delta v \cdot \nabla \Delta^{-1} g)$.

Next, we calculate $[\mathbf{D}_t, \Delta^{-1}]$. Let $g(t,x)$, $x \in \Omega_t$ be a smooth function and $\phi = \Delta^{-1}g$. From the first half of (1.10) where $\Delta f = 0$ was not used,

$$\mathbf{D}_t g = \mathbf{D}_t \Delta \phi = \Delta \mathbf{D}_t \phi - 2Dv \cdot D^2\phi - \Delta v \cdot \nabla \phi.$$

Since $\mathbf{D}_t \phi|_{\partial\Omega_t} = 0$, we obtain

$$\mathbf{D}_t \Delta^{-1} g = \Delta^{-1}\mathbf{D}_t g + \Delta^{-1}(2Dv \cdot D^2 \Delta^{-1} g + \Delta v \cdot \nabla \Delta^{-1} g) \tag{1.12}$$

- $[\mathbf{D}_t, \mathcal{N}]f = \nabla_N \Delta^{-1}(2Dv \cdot D^2 f_{\mathcal{H}} + \nabla f_{\mathcal{H}} \cdot \Delta v) - \nabla f_{\mathcal{H}} \cdot \nabla_N v - \nabla_{\nabla^\top f} v \cdot N$.

To calculate the commutator of $[\mathbf{D}_t, \mathcal{N}]$, from (1.1), (1.8) and (1.11), we have

$$\begin{aligned} \mathbf{D}_t(\nabla f_{\mathcal{H}} \cdot N) &= \nabla_N \mathbf{D}_t f_{\mathcal{H}} - \nabla f_{\mathcal{H}} \cdot \nabla_N v + \nabla f_{\mathcal{H}} \cdot \mathbf{D}_t N \\ &= \nabla_N[\mathcal{H}(\mathbf{D}_t f) + \Delta^{-1}(2Dv \cdot D^2 f_{\mathcal{H}} + \nabla f_{\mathcal{H}} \cdot \Delta v)] - \nabla f_{\mathcal{H}} \cdot \nabla_N v - \nabla_{\nabla^\top f} v \cdot N. \end{aligned}$$

Thus,

$$\mathbf{D}_t \mathcal{N}(f) = \mathcal{N}(\mathbf{D}_t f) + \nabla_N \Delta^{-1}(2Dv \cdot D^2 f_{\mathcal{H}} + \nabla f_{\mathcal{H}} \cdot \Delta v) - \nabla f_{\mathcal{H}} \cdot \nabla_N v - \nabla_{\nabla^\top f} v \cdot N. \tag{1.13}$$

- $[\Delta_{\partial\Omega_t}, \mathbf{D}_t]f = 2\mathcal{D}^2 f \cdot ((D^\top|_{T\partial\Omega_t})v) + \nabla^\top f \cdot \Delta_{\partial\Omega_t} v - \kappa \nabla_{\nabla^\top f} v \cdot N$.

In order to calculate the commutator $[\Delta_{\partial\Omega_t}, \mathbf{D}_t]$ at $x_0 \in \partial\Omega_{t_0}$, take an orthonormal frame $\{\tau_1, \ldots, \tau_{n-1}\}$ of $T_{x_0}\partial\Omega_t$. We first extend this to an orthonormal frame to $T_x\partial\Omega_{t_0}$ for all $x \in \partial\Omega_{t_0}$ close to x_0 by parallel transporting $\{\tau_1, \ldots, \tau_{n-1}\}$ along geodesics on $\partial\Omega_{t_0}$ starting from x_0. Parallel transporting them again along the material lines $x(t)$, we obtain an orthonormal frame $\{\tau_1, \ldots, \tau_{n-1}\}$ of $T_x\partial\Omega_t$ for all (t,x) near (t_0, x_0). From the standard Riemannian geometry, this orthonormal frame satisfies the property that, at (t_0, x_0), $\mathcal{D}\tau_j = 0$ and $[\tau_i, \tau_j] =$

$\mathcal{D}_{\tau_i}\tau_j - \mathcal{D}_{\tau_j}\tau_i = 0$, which will be used repeatedly. For any smooth function $f(t,x)$ defined on $\partial\Omega_t$, at (t_0,x_0),

$$\begin{aligned}\mathbf{D}_t\Delta_{\partial\Omega_t} f =& \mathbf{D}_t(\nabla_{\tau_j}\nabla_{\tau_j} f - \nabla_{\mathcal{D}_{\tau_j}\tau_j} f) = \nabla_{\tau_j}\mathbf{D}_t\nabla_{\tau_j} f + \nabla_{[\mathbf{D}_t,\tau_j]}\nabla_{\tau_j} f - \nabla_{[\mathbf{D}_t,\mathcal{D}_{\tau_j}\tau_j]} f\\ =&\nabla_{\tau_j}\nabla_{\tau_j}\mathbf{D}_t f + \nabla_{\tau_j}\nabla_{[\mathbf{D}_t,\tau_j]} f + \nabla_{[\mathbf{D}_t,\tau_j]}\nabla_{\tau_j} f - \nabla_{[\mathbf{D}_t,\mathcal{D}_{\tau_j}\tau_j]} f\\ =&\Delta_{\partial\Omega_t}\mathbf{D}_t f + 2\mathcal{D}^2 f(\tau_j,[\mathbf{D}_t,\tau_j]) + \nabla_{\mathcal{D}_{\tau_j}[\mathbf{D}_t,\tau_j]-[\mathbf{D}_t,\mathcal{D}_{\tau_j}\tau_j]} f.\end{aligned}$$

For any vector field $\tau(t,x) \in T_x\partial\Omega_t$, it is easy to see that $[\mathbf{D}_t,\tau] \in T_x\partial\Omega_t$ since (a) $\tau, \frac{\partial}{\partial t} + v \in T(\cup_t\partial\Omega_t) \Rightarrow [\mathbf{D}_t,\tau] \in T(\cup_t\partial\Omega_t)$ and (b) $[\mathbf{D}_t,\tau] = \mathbf{D}_t\tau - \nabla_\tau v$ does not have $\frac{\partial}{\partial t}$ component. Thus, $\mathcal{D}_{\tau_j}[\mathbf{D}_t,\tau_j] - [\mathbf{D}_t,\mathcal{D}_{\tau_j}\tau_j] \in T\partial\Omega_t$ and we can drop all the normal components in its calculation. Using $\mathcal{D}_{\tau_j}\tau_j = \nabla_{\tau_j}\tau_j + \kappa N$ and $\mathcal{D}\tau_j = 0$ at (t_0,x_0), we obtain at (t_0,x_0),

$$\begin{aligned}&\mathcal{D}_{\tau_j}[\mathbf{D}_t,\tau_j] - [\mathbf{D}_t,\mathcal{D}_{\tau_j}\tau_j] = \big(\nabla_{\tau_j}[\mathbf{D}_t,\tau_j] - \mathbf{D}_t(\nabla_{\tau_j}\tau_j + \kappa N)\big)^\top\\ =&(\nabla_{\tau_j}\mathbf{D}_t\tau_j - \nabla_{\tau_j}\nabla_{\tau_j} v - \mathbf{D}_t\nabla_{\tau_j}\tau_j - \kappa\mathbf{D}_t N)^\top = -(\Delta_{\partial\Omega_t} v)^\top + \kappa((Dv)^*(N)))^\top.\end{aligned}$$

Therefore, from (1.4),

$$\mathbf{D}_t\Delta_{\partial\Omega_t} f = \Delta_{\partial\Omega_t}\mathbf{D}_t f - 2\mathcal{D}^2 f\cdot((D^\top|_{T\partial\Omega_t})v) - \nabla^\top f\cdot\Delta_{\partial\Omega_t} v + \kappa\nabla_{\nabla^\top f} v\cdot N. \tag{1.14}$$

Calculation of $\mathbf{D}_t^2\kappa$. This calculation starts with formula (1.6). Since $\Pi : T\partial\Omega_t \to T\partial\Omega_t$ then $\Pi\cdot D^\top|_{T\partial\Omega_t} v = \Pi\cdot\nabla|_{T\partial\Omega_t} v$. Let $\{\tau_1,\ldots,\tau_{n-1}\}$ be an orthonormal frame which is the parallel transport of an orthonormal frame $T_{x_0}\partial\Omega_{t_0}$ along the material line $x(t)\in\partial\Omega_t$. From (1.1), (1.5), (1.8), and (1.6), we have at (t_0,x_0),

(1.15)

$$\begin{aligned}\mathbf{D}_t^2\kappa =& -\mathbf{D}_t\Delta_{\partial\Omega_t} v\cdot N - \Delta_{\partial\Omega_t} v\cdot\mathbf{D}_t N - 2(\mathbf{D}_t^\top(\Pi(\tau_i)))\cdot\nabla_{\tau_i} v - 2\Pi(\tau_i)\cdot\mathbf{D}_t(\nabla_{\tau_i} v)\\ =& -\mathbf{D}_t\Delta_{\partial\Omega_t} v\cdot N + \Delta_{\partial\Omega_t} v\cdot(Dv)^*(N)^\top + 2\mathcal{D}_{\tau_i}\left(((Dv)^*(N))^\top\right)\cdot\nabla_{\tau_i} v\\ &+ 2\Pi((\nabla_{\tau_i} v)^\top)\cdot\nabla_{\tau_i} v - 2\Pi(\tau_i)\cdot\nabla_{\mathbf{D}_t\tau_i} v - 2\Pi(\tau_i)\cdot\nabla_{\tau_i}\mathbf{D}_t v + 2\Pi(\tau_i)\cdot(Dv)^2(\tau_i)\\ =& -\mathbf{D}_t\Delta_{\partial\Omega_t} v\cdot N - 2\Pi\cdot(D^\top|_{T\partial\Omega_{t_0}}\mathbf{D}_t v) + \Delta_{\partial\Omega_t} v\cdot(Dv)^*(N)^\top + 2[\mathcal{D}\left(((Dv)^*(N))^\top\right)\\ &+ \Pi((D^\top|_{T\partial\Omega_{t_0}} v)^\top)]\cdot(D^\top|_{T\partial\Omega_{t_0}} v) + 2\Pi\cdot((Dv)^2|_{T\partial\Omega_t})^\top.\end{aligned}$$

To compute $\mathbf{D}_t\Delta_{\partial\Omega_t} v\cdot N$ from (1.14) we need the general formula

$$\mathcal{D}^2 f(\tau,\tau') = D^2 f(\tau,\tau') - (\Pi(\tau)\cdot\tau')\nabla_N f.$$

for any $\tau, \tau' \in T_{x_0}\partial\Omega_{t_0}$. Therefore,

(1.16)
$$-\mathbf{D}_t\Delta_{\partial\Omega_t}v\cdot N = -N\cdot\Delta_{\partial\Omega_t}\mathbf{D}_t v + 2N\cdot D^2 v(\tau_i, (\nabla_{\tau_i}v)^\top) - 2(\nabla_N v\cdot N)(\Pi\cdot(D^\top|_{T\partial\Omega_{t_0}}v)) + N\cdot\nabla v((\Delta_{\partial\Omega_t}v)^\top) - \kappa|(\nabla v)^*(N)^\top|^2.$$

When v and Ω_t satisfy the Euler's equation, the expression for $\mathbf{D}_t^2\kappa$ can be written as

$$\mathbf{D}_t^2\kappa = -N\cdot\Delta_{\partial\Omega_t}\mathbf{D}_t v + 2\epsilon^2\Pi\cdot(D^\top|_{T\partial\Omega}J) + r \tag{1.17}$$

where we signaled out the important terms in the above equation

2. Basic estimates for domains with limited regularity

In free boundary problems, it often happens that the moving domain Ω_t is of class H^s and moves with an H^{s_0} velocity field with $s_0 \le s$. Moreover, the estimates usually involve functions and vector fields defined on Ω_t and $\partial\Omega_t$. Therefore, in this section, we consider collections Λ of domains Ω which are H^{s_0} close to some reference domain and bounded in the H^s class in some sense to be defined rigorously. We will outline some basic estimates on functions defined on Ω and $\partial\Omega$ and some related operators. Through tedious derivation, these estimates will be guaranteed to be uniform for all $\Omega \in \Lambda$.

2.1. **Sobolev norms.** Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain, viewing $H^s(\Omega)$, $s \ge 0$, as a quotient space of $H^s(\mathbb{R}^n)$, define the norm

$$|g|_{H^s(\Omega)} = \inf\{|G|_{H^s(\mathbb{R}^n)} \ : \ G \in H^s(\mathbb{R}^n),\ G|_\Omega = g\}$$

where $|\cdot|_{H^s(\mathbb{R}^n)}$ is defined through the Fourier transform as

$$|G(x)|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(1+|\xi|^2)^s|\hat{G}(\xi)|^2 d\xi.$$

As usual, for $s \ge 0$, $H_0^s(\Omega)$ represents the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ and $H^{-s}(\Omega)$ is isometric to $(H_0^s(\Omega))^*$. It is important to note that with this definition of H^s norm the constants in Sobolev embedding ($H^s \to L^p$ or C^α) are independent of Ω. The relationship between this definition of H^s norm and the standard definition will be explored later on page 13.

$C^1 \cap H^2$ **boundary** $\partial\Omega$**.** To consider functions defined on $\partial\Omega$, let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain with $\partial\Omega$ of class $H^2 \cap C^1$. Consider the local graph coordinates of $\partial\Omega$ in orthonormal frames. When two coordinate charts of this type overlap, it is easy to verify that the transition map between these two local coordinate maps is also of $C^1 \cap H^2$. Therefore, on $\partial\Omega$, the definitions of spaces $C^1(\partial\Omega) \cap H^2(\partial\Omega)$ of scalar functions and $C^0(\partial\Omega) \cap H^1(\partial\Omega)$ of (k,l)-type tensors, though defined in local coordinates, are independent of the choice of local coordinates. The Christofell symbols and the usual geometric quantities of the hypersurface $\partial\Omega$, such as the second fundamental form and mean curvature are well-defined in $L^2(\partial\Omega)$ and the sectional curvature is in $L^1(\partial\Omega)$, for it is like the square of the second fundamental form. As these will be referred to later, we give the explicit formula in local coordinates here. Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame and $(x^1, \ldots, x^n)$ be the coordinates associated with this frame. Suppose Ω locally is given by $x^n > f(x^1, \ldots, x^{n-1})$ with $f \in H^2 \cap C^1$, then using $(x^1, \ldots, x^{n-1})$ as the local coordinates, we have

$$\text{The Christofell symbols } \Gamma^k_{ij} = \frac{\partial_k f \partial_{ij} f}{\sqrt{1+|\nabla f|^2}};$$

$$\text{The second fundamental form } \Pi(\frac{\partial}{\partial x^i}) \cdot \frac{\partial}{\partial x^j} = -\frac{\partial_{ij} f}{\sqrt{1+|\nabla f|^2}}$$

$$\text{Mean curvature } \kappa = -\partial_j \left(\frac{\partial_j f}{\sqrt{1+|\nabla f|^2}} \right) = -\frac{\Delta f}{(1+|\nabla f|^2)^{\frac{1}{2}}} + \frac{\partial_i f \partial_j f \partial_{ij} f}{(1+|\nabla f|^2)^{\frac{3}{2}}}; \tag{2.1}$$

$$\text{Sectional curvature } \mathcal{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \frac{\partial_{ii} f \partial_{jj} f - (\partial_{ij} f)^2}{1+|\nabla f|^2}$$

$$\text{Beltrami-Lapalace } \Delta_{\partial\Omega}\phi = \mathrm{tr}\mathcal{D}^2\phi = \mathcal{D} \cdot \nabla^\top \phi = \frac{1}{\sqrt{1+|\nabla f|^2}} \partial_i \left(g^{ij} \sqrt{1+|\nabla f|^2} \partial_j \phi \right);$$

where the matrix $(g^{ij}) = (\delta_{ij} + \partial_i f \partial_j f)^{-1}$. For a $C^0 \cap H^1$ tensor T of (k,l)-type, the covariant derivatives $\mathcal{D}T$ is a $(k, l+1)$-type tensor in L^2. For any (k,l) tensor T_1 and $(k, l+1)$ tensor T_2 in $C^0 \cap H^1$, one may verify, possibly through smooth approximations of $\partial\Omega$,

$$\int_{\partial\Omega} (\mathcal{D}T_1) \cdot T_2 \, dS = \int_{\partial\Omega} \mathrm{tr}\left(T_1 \cdot \mathcal{D}T_2(\cdot,\cdot)\right) \, dS.$$

where, on the above right side, $\mathcal{D}T_2(\cdot,\cdot)$ denotes the (k,l)-type tensor define by $\mathcal{D}T_2(X,Y)(\cdots) = (\mathcal{D}_X T_2)(Y, \cdots)$.

From this identity, for any L^2 tensor T, one can define $\mathcal{D}T$, in the distribution sense, as in the dual space of $C^0 \cap H^1$ tensors. It is straightforward to verify that, for any (k,l)-type tensor T in $C^0 \cap H^1$, we have

$$\int_{\partial\Omega} T \cdot \Delta_{\partial\Omega} T dS = -\int_{\partial\Omega} |\mathcal{D}T|^2 dS. \tag{2.2}$$

If $T, \mathcal{D}T \in C^0 \cap H^1$, we have

$$\int_{\partial\Omega} |\Delta_{\partial\Omega} T|^2 dS = \int_{\partial\Omega} |\mathcal{D}^2 T|^2 + \big[\mathcal{D}_{\mathcal{R}(X_{j_1}, X_{j_2})X_{j_1}} T - \mathcal{R}(X_{j_1}, X_{j_2})\, \mathcal{D}_{X_{j_1}} T\big] \cdot \mathcal{D}_{X_{j_2}} T - \frac{1}{2}|\mathcal{R}(X_{j_1}, X_{j_2})T|^2 dS. \tag{2.3}$$

Here $\{X_1, \ldots, X_{n-1}\}$ is any pointwise orthonormal frame of $\partial\Omega$ which always appears in the trace form resulting in the independence of the corresponding quantities of the choice of the frame. The curvature acts on T in the usual sense

$$\mathcal{R}(X,Y)T = \mathcal{D}_Y \mathcal{D}_X T - \mathcal{D}_X \mathcal{D}_Y T - \mathcal{D}_{[Y,X]} T = \mathcal{D}^2 T(Y,X) - \mathcal{D}^2 T(X,Y).$$

Though $\mathcal{R}(X,Y)T$ seems to contain derivatives of T, one may calculate

$$(\mathcal{R}(X,Y)T)(X_1, X_2, \ldots) = -T(\mathcal{R}(X,Y)X_1, X_2, \ldots) - T(X_1, \mathcal{R}(X,Y)X_2, \ldots) - \ldots.$$

So the dependence of $\mathcal{R}(X,Y)T$ on X, Y, T is only pointwise and $\mathcal{R}$ vanishes if T is a scalar function.

Since $I - \Delta_{\partial\Omega}$ is a positive self-adjoint operator on $L^2(\partial\Omega)$, for $\phi : \partial\Omega \to \mathbb{R}$ and $r \geq 0$, we define the Sobolev norm $|\cdot|_{H^r(\partial\Omega)}$ on the surface $\partial\Omega$ as

$$|\phi|^2_{H^r(\partial\Omega)} = \int_{\partial\Omega} |(I - \Delta_{\partial\Omega})^{\frac{r}{2}} \phi|^2 dS; \qquad |\cdot|_{L^2(\partial\Omega)} = |\cdot|_{H^0(\partial\Omega)}.$$

As usual, for $r \geq 0$, $|\cdot|_{H^{-r}(\partial\Omega)}$ coincides with $|\cdot|_{H^r(\partial\Omega)^*}$. One may note here, since the Christofell symbols Γ^k_{ij} are only in L^2, while $|T|_{H^1(\partial\Omega)} < \infty$ implies that $\mathcal{D}T \in L^2(\partial\Omega)$ from (2.2), it does not imply that T is in H^1 in local coordinates, except when T is of $(0,0)$-type, i.e. a scalar function. Similarly, from (2.3), $|T|_{H^2(\partial\Omega)} < \infty$ does not imply $\mathcal{D}^2 T \in L^2(\partial\Omega)$.

Remark. *When $n = 2$, $\partial\Omega$ is 1-dimension and $\Delta_{\partial\Omega} = \partial_{ss}$ where s is the arc length parameter, which is well defined if $\partial\Omega$ is in $W^{1,1}$. In fact, $\partial\Omega \in H^s$, $s > \frac{3}{2}$, is sufficient for the definitions of all the objects intrinsic in $\partial\Omega$.*

H^s **boundary** $\partial\Omega$**,** $s > \frac{n+1}{2}$**.** For the purpose of this paper, we assume $\partial\Omega \subset \mathbb{R}^n$ is in H^s with $s > \frac{n+1}{2}$. On the one hand, we defined the norm $|\cdot|_{H^r(\partial\Omega)}$ using the Beltrami-Lapalace $\Delta_{\partial\Omega}$ in the above. On the other hand, an obvious and traditional way to define the Sobolev space $H^r(\partial\Omega)$, $-s \leq r \leq s$ for scalar valued functions and $1-s \leq r \leq s-1$ for tensor valued functions, is through local coordinate coverings of $\partial\Omega$ and the definition of the Sobolev space

$H^r(\mathbb{R}^{n-1})$. From standard Sobolev inequalities, it is easy to see that the latter definition of the spaces $H^r(\partial\Omega)$ is actually independent of local coordinates and naturally induces a topology on $H^r(\partial\Omega)$. In particular, when $r \geq 0$ is an integer, straightforward calculation also shows that a function f (or tensor field) belongs to $H^r(\partial\Omega)$ if and only if $f, \mathcal{D}^r f \in L^2(\partial\Omega)$. In fact, we have

Proposition 2.1. *For $r \in [-s, s]$ ($r \in [1-s, s-1]$ for tensors), the norm $|\cdot|_{H^r(\partial\Omega)}$ is equivalent to the norm on $H^r(\partial\Omega)$ defined by using local coordinates.*

The proof of this proposition follows from the standard elliptic estimates using the local coordinates along with interpolation. In particular, when s is an integer, one may also prove it geometrically. In fact, the proposition clearly holds for $r = 1$ and $r = 2$ due to (2.2) and (2.3) and Sobolev inequalities. When r is an integer and $r \in [3, s]$ ($r \in [3, s-1]$ for tensors), the proposition can be proved by using the following identity

$$([\Delta_{\partial\Omega}, \mathcal{D}]T)(X) = \mathcal{R}(X, X_j)\mathcal{D}_{X_j}\, T + (\mathcal{D}_{X_j}\mathcal{R})(X, X_j)T + (\mathcal{R}(X, X_j)\mathcal{D}\, T)(X_j). \tag{2.4}$$

Finally, for non-integer or negative r, the proposition follows from interpolation and duality. Another implication of (2.4) is that $\mathcal{D} : H^r(\partial\Omega) \to H^{r-1}(\partial\Omega)$ is bounded and $|\cdot|_{L(H^r(\partial\Omega), H^{r-1}(\partial\Omega))}$ depends only on $\mathcal{R}$ and its derivatives.

It is well known that the regularity of $\partial\Omega$ can be determined from the regularity of its mean curvature κ.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain such that $\partial\Omega \in H^{s_0}$, $s_0 > \frac{n+1}{2}$. Suppose $|\kappa|_{H^{s-2}(\partial\Omega)} < \infty$ with $s > s_0$, then $\partial\Omega \in H^s$.*

Proposition 2.2 can be proved by using local coordinates and standard quasilinear estimates. Another proof can be based on the following identity which is also used in a priori estimates. Intuitively, let $e : \partial\Omega \to \mathbb{R}^n$ be the imbedding, then $\Pi = -N \cdot \mathcal{D}^2 e$ and $\kappa = -N \cdot \Delta_{\partial\Omega}\, e$, where Π is viewed as a symmetric quadratic form. Thus it is expected that the difference $\Delta_{\partial\Omega}\Pi - \mathcal{D}^2\kappa$ should be of lower order terms only. In fact,

$$-\Delta_{\partial\Omega}\Pi = -\mathcal{D}^2\kappa + |\Pi|^2\Pi - \kappa\Pi^2. \tag{2.5}$$

To prove the identity, at any $x \in \partial\Omega$, let λ_i, $i = 1, \ldots, n-1$, be the eigenvalues of $\Pi(x)$ and X_i be the associated eigenvectors which form an orthonormal frame of $T_x\partial\Omega$. Parallel transport this frame to every base point in a neighborhood of x on $\partial\Omega$ along the radial geodesics emitting from x. From the construction, we have $\mathcal{D}_{X_i}X_j = 0$, $[X_i, X_j] = 0$, and $\Pi(X_i) = \lambda_i X_i$ at x. For any $X = a^j X_j$ with constants $a^1, \ldots, a^{n-1}$, at x,

$$\begin{aligned}(\Delta_{\partial\Omega}\Pi)(X,X) =&(\mathcal{D}_{X_i}\mathcal{D}_{X_i}\Pi)(X,X) = \nabla_{X_i}((\mathcal{D}_{X_i}\Pi)(X,X)) = \nabla_{X_i}((\mathcal{D}_{X}\Pi)(X_i,X))\\ =&(\mathcal{D}_{X_i}\mathcal{D}_{X}\Pi)(X_i,X) = (\mathcal{D}_{X}\mathcal{D}_{X_i}\Pi)(X,X_i) + (\mathcal{R}(X,X_i)\Pi)(X_i,X)\\ =&\nabla_X((\mathcal{D}_{X_i}\Pi)(X,X_i)) + \Pi(\mathcal{R}(X_i,X)X_i,X) + \Pi(X_i,\mathcal{R}(X_i,X)X).\end{aligned}$$

For the first term at x, from the construction of our special frame,

$$\nabla_X((\mathcal{D}_{X_i}\Pi)(X,X_i)) = \nabla_X((\mathcal{D}_{X}\Pi)(X_i,X_i)) = \nabla_X\nabla_X\kappa - 2\Pi(X_i,\mathcal{D}_X\mathcal{D}_X X_i) = \mathcal{D}^2\kappa(X,X).$$

To calculate the remaining two terms, one may substitute $X = a^j X_j$ and use $\Pi(X_j) = \lambda_j X_j$, the symmetry of $\mathcal{R}$, and the following calculation

$$\begin{aligned}4\mathcal{R}(X_i,X_{j_1})X_i\cdot X_{j_2} =&\mathcal{R}(X_i,X_{j_1}+X_{j_2})X_i\cdot(X_{j_1}+X_{j_2}) - \mathcal{R}(X_i,X_{j_1}-X_{j_2})X_i\cdot(X_{j_1}-X_{j_2})\\ =&\Pi(X_i,X_i)\Pi(X_{j_1}+X_{j_2},X_{j_1}+X_{j_2}) - \Pi(X_i,X_{j_1}+X_{j_2})^2\\ &- \Pi(X_i,X_i)\Pi(X_{j_1}-X_{j_2},X_{j_1}-X_{j_2}) + \Pi(X_i,X_{j_1}-X_{j_2})^2\\ =&4\delta_{j_1j_2}\lambda_i\lambda_{j_1} - 4\delta_{ij_1}\delta_{ij_2}\lambda_i^2.\end{aligned}$$

Equality (2.5) follows consequently.

H^{s_0} **neighborhoods of domains,** $s_0 > \frac{n+1}{2}$**.** Given a domain Ω_* with $\partial\Omega_*$ in H^{s_0}, we need to consider the following type of H^{s_0} neighborhoods of Ω_*, a bounded connected domain in $\mathbb{R}^n$, which are bounded in H^s for some $s \geq s_0$.

Definition 2.1. *Let $\Lambda = \Lambda(\Omega_*, s_0, s, L, \delta)$ be the collection of all domains Ω satisfying*

(A1) *there exists a diffeomorphism $F : \partial\Omega_* \to \partial\Omega \subset \mathbb{R}^n$, so that $|F - id_{\partial\Omega_*}|_{H^{s_0}(\partial\Omega_*)} < \delta$;*
(A2) *the mean curvature κ of $\partial\Omega$ satisfies $|\kappa|_{H^{s-2}(\partial\Omega)} < L$.*

From Proposition 2.2, every $\Omega \in \Lambda$ is in H^s. Given Ω_* and sufficiently small $\delta > 0$, in the following, we will derive some estimates with bounds C uniform in $\Omega \in \Lambda$. Since $\partial\Omega_*$ is compact, for any $\sigma > 0$, there exist $x_i \in \mathbb{R}^n$ and $d, d_i \in (0, \frac{1}{2}]$, $i = 1, \ldots, m$, (B1) $B(\partial\Omega_*, d) \subset \cup_{i=1}^m R_i(d_i)$ where each $R_i(\cdot) = \tilde{R}_i(\cdot) \times I_i(\cdot) \subset \mathbb{R}^n$ with $\tilde{R}_i(\cdot)$ and $I_i(\cdot)$ being an open $(n-1)$-dimensional disk and an open perpendicular segment in $\mathbb{R}^n$, both centered at x_i and of the given radius and half length, respectively;

(B2) For each i, $z = (z_1, \ldots, z_{n-1}, z_n) = (\tilde{z}, z_n)$ being an Euclidean coordinate system on $\tilde{R}_i(\cdot) \times I_i(\cdot)$, there exists an H^s function $f_{*i} : \tilde{R}_i(2d_i) \to I_i$, so that

$$|f_{*i}|_{C^0} < \sigma d_i, \qquad |Df_{*i}|_{C^0} < \sigma, \qquad \text{and} \qquad \Omega_* \cap R_i(2d_i) = \{z^n > f_{*i}(\tilde{z})\}. \tag{2.6}$$

For any $\sigma > 0$ with a fixed coordinate covering $\{R_i(d_i)\}_{i=1}^m$ of $\partial\Omega_*$ of the above type, it is clear that, when $\delta > 0$ is sufficiently small, $\{R_i(d_i)\}_{i=1}^m$ is still a coordinate covering of any $\partial\Omega \in \Lambda$ satisfying (B1) and (B2) with coordinate functions $\{f_i \in H^s\}_{i=1}^m$. This will provide us some technical convenience in deriving estimates uniform in $\Omega \in \Lambda$. The following proposition is a refinement of Proposition 2.2.

Proposition 2.3. *Given Ω_*, there exists $\delta > 0$ such that, for any $L > 0$, there exists $C > 0$ such that the second fundamental form of any $\Omega \in \Lambda$ satisfies*

$$|\Pi|_{H^{s-2}(\partial\Omega)} < C.$$

Proof. The proof follows simply from the standard elliptic estimates and we will only give a sketch. With $\partial\Omega \in H^s$ due to Proposition 2.2, we will use the above coordinate covering $\{R_i(d_i)\}_{i=1}^m$ and the coordinate functions $\{f_i \in H^s(\tilde{R}_i(2d_i))\}_{i=1}^m$, whose $H^{s_0}(\tilde{R}_i(2d_i))$ norms are uniformly bounded in i and Ω. Let $\gamma : [0, +\infty) \to [0, 1]$ be a smooth cut-off function supported on $[0, \frac{3}{2}]$ and $\gamma|_{[0,\frac{5}{4}]} \equiv 1$. On each $\tilde{R}_i(2d_i)$, let

$$\gamma_i(\tilde{z}) = \gamma(\frac{|\tilde{z}|}{d_i}), \qquad \kappa_i(\tilde{z}) = \gamma_i(\tilde{z})\kappa(\tilde{z}, f_i(\tilde{z})), \qquad g_i = \gamma_i f_i,$$

where κ is the mean curvature of $\partial\Omega$. It is clear from the definition of Λ that $|\kappa_i|_{H^{s_1-2}(\tilde{R}_i(d_i))}$ is bounded uniformly in i and Ω for $s_1 = \min\{s_0+2, s\}$. From the mean curvature formula (2.1),

$$\begin{aligned} -\Delta g_i + \frac{\partial_{j_1} f_i \partial_{j_2} f_i}{1 + |\nabla f_i|^2} \partial_{j_1 j_2} g_i =& (1 + |\nabla f_i|^2)^{\frac{1}{2}} \kappa_i - \Delta\gamma_i f_i - 2D\gamma_i \cdot Df_i \\ &+ \frac{\partial_{j_1} f_i \partial_{j_2} f_i}{1 + |\nabla f_i|^2} (\partial_{j_1 j_2}\gamma_i f_i + \partial_{j_1}\gamma_i \partial_{j_2} f_i + \partial_{j_2}\gamma_i \partial_{j_1} f_i) \end{aligned}$$

Since γ is supported on $[0, \frac{3}{2}]$, without loss of generality, we may treat f_i as compactly supported on the ball of radius $\frac{7d_i}{4}$ because f_i can always be replaced by $\tilde{\gamma}(\frac{|\tilde{z}|}{d_i}) f_i(\tilde{z})$ where $\tilde{\gamma}$ is a cut-off function supported on $[0, \frac{7}{4}]$ and $\tilde{\gamma}|_{[0,\frac{3}{2}]} = 1$. By partition of the unity and the Inverse Function Theorem, $(f_1, \ldots, f_m)$ can be expressed by $g = (g_1, \ldots, g_m)$ with the same regularity and similar estimates. Thus, dividing both sides of the above equation by $(1 + |\nabla f_i|^2)^{\frac{1}{2}}$, it can be rewritten as

$$-\Delta g + A_{j_1 j_2}(x, g, \partial g)\partial_{j_1 j_2} g = \kappa + G(x, g, \partial g),$$

where $A_{j_1 j_2}$, G_1, G_2 are smooth in their arguments and $A_{j_1 j_2} \leq C\sigma^2$ so the left side is uniformly elliptic. In this form, the estimate on $|g|_{H^{s_1}(\mathbb{R}^{n-1})}$ uniform in Ω is obtained following the standard theory of quasilinear elliptic equations. If $s_1 < s$, this procedure can be carried out again with s_0 replaced by $s_1 = s_0 + 2$. Thus the desired uniform estimates on f_i in H^s follow by repeating this procedure. □

Remark. *1) One can also prove Proposition 2.3 based on* (2.5).
2) A more careful estimate can be obtained for Π *in terms of* κ*.*

Using (2.2), (2.3), (2.4), and the above uniform estimate on Π, which implies the uniform estimate on the curvature $\mathcal{R}$, it is easy to prove that, for any tensor $T \in H^r(\partial\Omega)$, $r \in [2 - s, s - 1]$ ($r \in [1 - s, s]$ for scalars), we have

$$|\mathcal{D}T|_{H^{r-1}(\partial\Omega)} \leq C|T|_{H^r(\partial\Omega)} \tag{2.7}$$

for some C uniform in $\Omega \in \Lambda$.

From the uniform estimates on those (uniformly fixed) local coordinates derived in the above proof, it is also clear that the constants in the Sobolev inequalities (e.g. $|\cdot|_{H^s(\partial\Omega)}$ to $L^p(\partial\Omega)$ or $C^\alpha(\partial\Omega)$ for $s \leq k$) are uniform in $\Omega \in \Lambda$. The two most used inequalities in this paper are for $f \in H^{s_1}(\partial\Omega)$ and $g \in H^{s_2}(\partial\Omega)$, $s_1 \leq s_2$,

$$|fg|_{H^{s_1+s_2-\frac{n-1}{2}}(\partial\Omega)} \leq C|f|_{H^{s_1}(\partial\Omega)}|g|_{H^{s_2}(\partial\Omega)}, \qquad \text{if } s_2 < \frac{n-1}{2} \text{ and } 0 < s_1 + s_2$$

$$|fg|_{H^{s_1}(\partial\Omega)} \leq C|f|_{H^{s_1}(\partial\Omega)}|g|_{H^{s_2}(\partial\Omega)}, \qquad \text{if } s_2 > \frac{n-1}{2} \text{ and } 0 \leq s_1 + s_2.$$

Similar inequalities hold for f and g defined in Ω.

2.2. **Dirichlet-Neumann operator.** Given Ω_*, in order to study the Dirichlet-Neumann operator for domains $\Omega \in \Lambda \triangleq \Lambda(\Omega_*, s_0, s, L, \delta)$, we need to first construct local coordinate maps on each $R_i(2d_i)$ for each Ω, which flatten $\partial\Omega$ and have estimates uniform in $\Omega \in \Lambda$, based on the above coordinates functions of $\partial\Omega$.

Local coordinates and partition of the unit. From Proposition 2.3, $\partial\Omega \cap R_i(2d_i)$ is represented as the graph of an H^s function $f_i : \tilde{R}_i(2d_i) \to \mathbb{R}$. Let $\phi = \gamma_i f_i$ with γ_i defined in the previous proof. A standard way to extend ϕ to a function $\Phi \in H^{s+\frac{1}{2}}(\mathbb{R}^n)$ is through the Fourier transform with an appropriate constant a:

$$\hat{\Phi}(\xi^1, \ldots, \xi^n) = a \frac{(1 + (\xi^1)^2 + \ldots + (\xi^{n-1})^2)^s}{(1 + (\xi^1)^2 + \ldots + (\xi^n)^2)^{s+\frac{1}{2}}} \hat{\phi}(\xi^1, \ldots, \xi^{n-1}) \tag{2.8}$$

Since $s + \frac{1}{2} > \frac{n}{2} + 1$ and Φ is bounded in $H^{s+\frac{1}{2}}$ uniformly in i and Ω, $|D\Phi|_{C^0}$ is also uniformly bounded. Therefore, there exists $b > 0$ so that

$$H_i(z^1, \ldots, z^n) = (z^1, \ldots, z^{n-1}, bz^n + \Phi(z^1, \ldots, z^n)) \tag{2.9}$$

is a diffeomorphism so that $|H_i|_{H^{s+\frac{1}{2}}(\mathbb{R}^n)}$ and $|(H_i)^{-1}|_{H^{s+\frac{1}{2}}(\mathbb{R}^n)}$ are bounded uniformly in i and Ω. Let $G_i = (H_i)^{-1}$ and $g_i(z)$ be the n-the component of G_i, then $|g_i|_{H^{s+\frac{1}{2}}(\mathbb{R}^n)}$, $\partial_{z_n} g_i$, and $(\partial_{z_n} g_i)^{-1}$ are bounded uniformly in i and Ω. Obviously, there exists a uniform $\delta_* > 0$ so that

$$(\tilde{R}_i(\frac{5}{4}d_i) \times I_i(\frac{5}{4}\delta_* d_i)) \cap \Omega = \tilde{R}_i(\frac{5}{4}d_i) \times I_i(\frac{5}{4}\delta_* d_i)) \cap \{g_i > 0\}.$$

Based on the local coordinate maps, we can construct partition of the unit satisfying estimates uniform in Ω if δ is small. In fact, take $\gamma, \xi : C^\infty([0, +\infty), [0, 1])$ so that $\mathrm{supp}(\gamma) \subset [0, \frac{5}{4}]$, $\gamma|_{[0, \frac{9}{8}]} \equiv 1$, $\xi(r) = r$ for $r \geq \frac{2}{3}$, and $\xi|_{[0, \frac{1}{3}]} \equiv \frac{1}{3}$. Define

$$\tilde{\gamma}_{*i}(z) = \gamma(\frac{|\tilde{z}|}{d_i})\gamma(\frac{|z_n|}{\delta_* d_i}), \quad \eta = \xi \circ \Sigma_{i=1}^m (\tilde{\gamma}_{*i} \circ G_i), \quad \gamma_{*i} = \frac{\tilde{\gamma}_{*i} \circ G_i}{\eta}, \quad \gamma_{*0} = (1 - \Sigma_{i=1}^m \gamma_{*i})\chi(\Omega).$$

It is straight forward to verify that $\gamma_{*0}, \gamma_{*1}, \ldots, \gamma_{*m} \in H^{s+\frac{1}{2}}(\mathbb{R}^n, [0, 1])$ satisfy

$$|\gamma_{*i}|_{H^{s+\frac{1}{2}}(\mathbb{R}^n)} \leq C, \qquad \mathrm{supp}(\gamma_{*i}) \subset \tilde{R}_i(\frac{5}{4}d_i) \times I_i(\frac{5}{4}\delta_* d_i)),$$

for $i = 1, \ldots, m$, and $(\Sigma_{i=0}^m \gamma_{*i})|_\Omega \equiv 1$.

Remark. *Using the above local coordinates and partition of unity we can establish the equivalence of the standard H^ℓ norm and the norm given in definition 2.1 for integer $\ell \in (-s - \frac{1}{2}, s + \frac{1}{2})$. The ratio of the two norms is bounded above and below by two constants depending only on Λ.*

Trace and Harmonic extension. Let $s_1 \in (\frac{1}{2}, s+\frac{1}{2}]$. Using the partition of the unit and the above local coordinates, it is straight forward to obtain the trace operator estimate

$$|\,(\Psi|_{\partial\Omega})\,|_{H^{s_1-\frac{1}{2}}(\partial\Omega)} \le C|\Psi|_{H^{s_1}(\Omega)} \tag{2.10}$$

for any $\Psi \in H^{s_1}(\Omega)$ where $C > 0$ is uniform in $\Omega \in \Lambda$.

In order to obtain the estimate on the Harmonic extension operator, we first construct an extension for convenience. Let $s_2 \in (0, s]$ and $\psi \in H^{s_2}(\partial\Omega)$. Take the same auxiliary functions γ and ξ used above. For each $1 \le i \le m$, let $\phi_i(\tilde z) = \gamma(\frac{|\tilde z|}{d_i})\psi(H_i(\tilde z, 0))$ and $\Phi_i(z)$ be the extension of ϕ_i, defined in the way of (2.8). Let

$$\tilde\Phi_i(z) = \Phi_i(z)\gamma(\frac{|\tilde z|}{d_i})\gamma(\frac{|z_n|}{\delta_* d_i}), \qquad \Psi_1 = \Sigma_{i=1}^m \tilde\Phi_i \circ G_i$$

where Ψ_1 can be viewed as a function defined on $\mathbb{R}^n \supset \Omega$. Let

$$\eta_i(z) = \gamma(\frac{|\tilde z|}{d_i})^2\gamma(\frac{|z_n|}{\delta_* d_i}), \qquad \eta = \xi \circ \Sigma_{i=1}^m(\eta_i \circ G_i), \qquad \Psi = \frac{\Psi_1}{\eta}.$$

It is easy to verify that $\Psi \in H^{s_2+\frac{1}{2}}(\mathbb{R}^n)$ is an extension of $\psi \in H^{s_2}(\partial\Omega)$ satisfying the estimate

$$|\Psi|_{H^{s_2+\frac{1}{2}}(\mathcal{R}^n)} \le C|\psi|_{H^{s_2}(\partial\Omega)} \tag{2.11}$$

with C uniform in $\Omega \in \Lambda$.

Using the partition of the unit and the local coordinates we constructed above and following the standard procedure, we have

Lemma 2.4. *There exists $C > 0$ which depends only on the set Λ so that, for $s_1 \in [\frac{1}{2}, s]$*

$$|\Delta^{-1}|_{L(H^{s_1-\frac{3}{2}}(\Omega), H^{s_1+\frac{1}{2}}(\Omega))} + |\mathcal{H}|_{L(H^{s_1}(\Omega), H^{s_1+\frac{1}{2}}(\Omega))} \le C.$$

Dirichlet-Neumann operator. Following from the above estimate, the Dirichlet-Neumann operator $\mathcal{N} : H^{s_1}(\partial\Omega) \to H^{s_1-1}(\partial\Omega)$ can be defined and it has a uniform bound for $s_1 \in (1, s]$. In fact, we can extend $\mathcal{N}$ into a weaker form defined on $H^{s_1}(\partial\Omega)$ for $s_1 \ge \frac{1}{2}$. Given $f \in H^{\frac{1}{2}}(\partial\Omega)$, define $\mathcal{N}(f) \in H^{-\frac{1}{2}}(\partial\Omega)$ as

$$< \psi, \mathcal{N}(f) >= \int_\Omega \nabla f_{\mathcal{H}} \cdot \nabla \psi_{\mathcal{H}} dx$$

for any $\psi \in H^{\frac{1}{2}}(\partial\Omega)$. It is easy to prove that

1) $\mathcal{N}$ is self-adjoint in $L^2(\partial\Omega)$ with compact resolvent;

2) the kernel $\ker(\mathcal{N}) = \{\text{const}\}$;

3) $C|f|_{H^{\frac{1}{2}}(\partial\Omega)} \ge |\mathcal{N}(f)|_{H^{-\frac{1}{2}}(\partial\Omega)} \ge \frac{1}{C}|f|_{H^{\frac{1}{2}}(\partial\Omega)}$ for any f satisfying $\int_{\partial\Omega} f dS = 0$.

The first inequality of 3) follows from the uniform bound on $\mathcal{H}$. In order to prove the second inequality in (3), one notices that

$$|f|_{H^{\frac{1}{2}}(\partial\Omega)}|\mathcal{N}(f)|_{H^{-\frac{1}{2}}(\partial\Omega)} \ge |< f, \mathcal{N}(f) >| = \int_\Omega |\nabla\mathcal{H}(f)|^2 dx.$$

From the estimate of the trace operator, we only need, for any f satisfying $\int_{\partial\Omega} f dS = 0$,

$$|\mathcal{H}(f)|_{L^2(\Omega)} \le C|\nabla\mathcal{H}(f)|_{L^2(\Omega)}$$

with a constant C uniform in f and Ω. This inequality can be proved by a compactness argument. Thus, by duality and interpolation, $\mathcal{N}$ can be extended to $H^{s_1}(\partial\Omega)$ for all $s \in [1-s, s]$ and $|\mathcal{N}|_{L(H^{s_1}(\partial\Omega), H^{s_1-1}(\partial\Omega))}$ is bounded uniformly in Ω. Moreover, for $f \in H^s(\partial\Omega)$ with $\int_{\partial\Omega} f dS = 0$, we can obtain $|f|_{H^s(\partial\Omega)} \le C|\mathcal{N}(f)|_{H^{s-1}(\partial\Omega)}$ with C uniform in Ω. The proof is simply the elliptic estimate under the Neumann boundary condition – very much similar to the derivation of the harmonic extension estimate, except in the first step, instead of using (2.8), we need to construct F with $|F|_{H^{s+\frac{1}{2}}(\Omega)} \le C|\mathcal{N}(f)|_{H^{s-1}(\partial\Omega)}$ and $\nabla_N F = \mathcal{N}(f)$ on $\partial\Omega$, by using a slightly different formula of the same fashion. Therefore, from interpolation, we have, for any $s_1 \in [\frac{1}{2}, s]$,

$$|f|_{H^{s_1}(\partial\Omega)} \le C|\mathcal{N}(f)|_{H^{s_1-1}(\partial\Omega)}, \qquad \text{if } \int_{\partial\Omega} f dS = 0$$

with C uniform in Ω. moreover, this inequality holds for $s_1 \in [1-s, s]$ by duality. Based on these estimates, we can use $I + \mathcal{N}$ to define the Sobolev norms which are equivalent to those defined by using $I - \Delta_{\partial\Omega}$ uniformly in Ω, i.e.

Proposition 2.5. *For $s_1 \in [-s, s]$, the norms on $H^{s_1}(\partial\Omega)$ defined by interpolating $I - \Delta^T$ and $I + \mathcal{N}$ are equivalent, i.e.*

$$\frac{1}{C}(I - \Delta_{\partial\Omega})^{\frac{s_1}{2}} \le (I + \mathcal{N})^{s_1} \le C(I - \Delta_{\partial\Omega})^{\frac{s_1}{2}}$$

with C uniform in $\Omega \in \Lambda$.

Furthermore, for $s_1 \in [-s, s-1]$,

$$\mathcal{N}^{-1} : \dot{H}^{s_1}(\partial\Omega) \to \dot{H}^{s_1+1}(\partial\Omega), \qquad \dot{H}^{s_1}(\partial\Omega) = \{f \in H^{s_1}(\partial\Omega) \mid \int_{\partial\Omega} f dS = 0\}$$

is well defined and bounded uniformly in Ω. $\mathcal{N}^{-1}$ defined on $\cdot H^{-\frac{1}{2}}(\partial\Omega)$ induces the solvability of the Lapalace equation with Neumann boundary data given in $H^{-\frac{1}{2}}(\partial\Omega)$.

To demonstrate that $\mathcal{N}$ behaves like differentiation, we give the following "product rule". Given functions f and g defined on $\partial\Omega$. Since

$$f_{\mathcal{H}}g_{\mathcal{H}} - \Delta^{-1}\Delta(f_{\mathcal{H}}g_{\mathcal{H}}) = \mathcal{H}(f_{\mathcal{H}}g_{\mathcal{H}}|_{\partial\Omega}) = \mathcal{H}(fg) \qquad \text{in } \Omega,$$

we obtain

$$\mathcal{N}(fg) = f\mathcal{N}(g) + g\mathcal{N}(f) - 2\nabla_N \Delta^{-1}(\nabla f_{\mathcal{H}} \cdot \nabla g_{\mathcal{H}}). \tag{2.12}$$

Since $\mathcal{N}$ is like differentiation, coordinate independent, and self-adjoint, appearing naturally in the Euler's equation, it is sometimes convenient to express the Sobolev norms on $\partial\Omega$ by $\mathcal{N}$.

Relationship between $\mathcal{N}$ and $\Delta_{\partial\Omega}$. In addition to just the comparison between the norms of $\Delta_{\partial\Omega}$ and $\mathcal{N}$, we will prove that $\mathcal{N}$ is simply equal to $(-\Delta_{\partial\Omega})^{\frac{1}{2}}$ plus lower order terms. This improves the previous estimates and makes the estimates of some Sobolev norms using $\mathcal{N}$ more convenient. From the identity

$$\Delta\psi = \Delta_{\partial\Omega}\psi + \kappa\nabla_N\psi + D^2\psi(N,N) \qquad x \in \partial\Omega \tag{2.13}$$

for any smooth function ψ on Ω. Recall that $N_{\mathcal{H}}(x)$ and $\kappa_{\mathcal{H}}(x)$, $x \in \Omega$, denote the harmonic extension of the unit outward normal vector and the mean curvature of $\partial\Omega$. Given smooth $f : \partial\Omega \to \mathbb{R}$, at any $x \in \partial\Omega$,

$$\begin{aligned} D^2 f_{\mathcal{H}}(N,N) =& \nabla_N \nabla_{N_{\mathcal{H}}} f_{\mathcal{H}} - \nabla f_{\mathcal{H}} \cdot \nabla_N N_{\mathcal{H}} \\ =& \nabla_N \big(\mathcal{H}((\nabla_N f_{\mathcal{H}})|_{\partial\Omega}) + (-\Delta)^{-1}(-\Delta)(\nabla_{N\mathcal{H}} f_{\mathcal{H}})\big) - \mathcal{N}(N)\cdot(\mathcal{N}(f)N + \nabla^{\top} f) \\ =& \mathcal{N}^2(f) - 2\nabla_N(-\Delta)^{-1}(DN_{\mathcal{H}} \cdot D^2 f_{\mathcal{H}}) - \mathcal{N}(N)\cdot(\mathcal{N}(f)N + \nabla^{\top} f) \end{aligned}$$

which implies

$$(-\Delta_{\partial\Omega} - \mathcal{N}^2)f = \kappa\mathcal{N}(f) - 2\nabla_N(-\Delta)^{-1}(DN_{\mathcal{H}} \cdot D^2 f_{\mathcal{H}}) - \mathcal{N}(N)\cdot(\mathcal{N}(f)N + \nabla^{\top} f). \tag{2.14}$$

Proposition 2.6. *1) For $s > \frac{n+3}{2}$, there exists $C > 0$ uniform in $\Omega \in \Lambda$ such that we have*

$$|\Delta_{\partial\Omega} + \mathcal{N}^2|_{L(H^{s'}(\partial\Omega), H^{s'-1}(\partial\Omega))} \le C, \qquad s' \in [2-s, s-1].$$

2) For $s \in (\frac{n+1}{2}, \frac{n+3}{2})$ and $s > 2$, there exists $C > 0$ uniform in $\Omega \in \Lambda$ such that we have

$$|\Delta_{\partial\Omega} + \mathcal{N}^2|_{L(H^{s'}(\partial\Omega), H^{s'-\frac{n+5}{2}+s}(\partial\Omega))} \leq C, \qquad s' \in (2-s, \frac{n+1}{2}).$$

Proof. For $s' > 1$ or $s' > \frac{n+5}{2} - s$ in each case, the above inequalities follow directly from (2.14) and the estimates on $\mathcal{H}$ and $\mathcal{N}$. Thus, by duality and interpolation, we only need to consider $s' = \frac{1}{2}$ or $s' = \frac{1}{2}(\frac{n+5}{2} - s)$ in each case, respectively. Let $f, g : \partial\Omega \to \mathbb{R}$ be smooth and harmonically extend into Ω. Equality (2.14) yields

$$\int_{\partial\Omega} g(-\Delta_{\partial\Omega} - \mathcal{N}^2) f dS = \int_{\partial\Omega} \kappa g \mathcal{N}(f) + g\mathcal{N}(N) \cdot (\mathcal{N}(f)N + \nabla^\top f) dS - 2\int_\Omega DN_{\mathcal{H}}(\nabla g_{\mathcal{H}}) \cdot \nabla f_{\mathcal{H}} dx,$$

which is sufficient to establish the estimate. □

Corollary. *The proposition implies the commutator estimates*

$$|[\Delta_{\partial\Omega}, \mathcal{N}]|_{L((H^{s'}(\partial\Omega), H^{s'-2}(\partial\Omega))} \leq C, \qquad s' \in [3-s, s-1] \tag{2.15}$$

if $s > \frac{n+3}{2}$ and

$$|[\Delta_{\partial\Omega}, \mathcal{N}]|_{L((H^{s'}(\partial\Omega), H^{s'-\frac{n+7}{2}+s}(\partial\Omega))} \leq C, \qquad s' \in (3-s, \frac{n+1}{2}), \tag{2.16}$$

if $s \in (\frac{n+1}{2}, \frac{n+3}{2})$ and $s > 2$.

We need the following abstract result for a more careful estimate on $\mathcal{N}$.

Proposition 2.7. *Let X be a Hilbert space and A and B be (possibly unbounded) self-adjoint positive operators on X so that $A^{-1}B$ and AB^{-1} are bounded. Suppose $K = A^2 - B^2$ satisfies that $KB^{-\alpha}$ is bounded with $\alpha \in [0, 2)$, then $(A - B)B^{1-\alpha}$ is bounded.*

Proof. Let $R = A - B$. Calculating $(B + R)^2 = B^2 + K$, we obtain

$$-BR - RB = R^2 - K,$$

which implies

$$\frac{d}{dt}(e^{-Bt} R e^{-Bt}) = e^{-Bt}(R^2 - K)e^{-Bt} \geq -e^{-Bt} K e^{-Bt} \geq -Ce^{-Bt} B^\alpha e^{-Bt}.$$

Therefore,

$$R \leq C_1 \int_0^\infty e^{-Bt} B^\alpha e^{-Bt} dt = \frac{C_1}{2} B^{\alpha-1}$$

Calculating $A^2 = (A-R)^2 + K$ with a similar procedure, we obtain

$$R \geq -C_2 \int_0^\infty e^{-At} A^\alpha e^{-At} dt = -\frac{C_2}{2} A^{\alpha-1}.$$

Thus, the conclusion follows. □

From Proposition 2.6 and Proposition 2.7, we obtain

Theorem 2.1. *There exist $C > 0$, which depends only on the set Λ such that if $s > \frac{n+3}{2}$*

$$|(-\Delta_{\partial\Omega})^{\frac{1}{2}} - \mathcal{N}|_{L(H^{s'}(\partial\Omega))} \leq C, \qquad s' \in [1-s, s-1]$$

and if $s > 2$ and $s \in (\frac{n+1}{2}, \frac{n+3}{2})$, for $\alpha = \frac{n+5}{2} - s$,

$$|(-\Delta_{\partial\Omega})^{\frac{1}{2}} - \mathcal{N}|_{L(H^{s'}(\partial\Omega), H^{s'-\alpha+1}(\partial\Omega))} \leq C, \qquad s' \in (1-s, \frac{n+1}{2}). \tag{2.17}$$

Proof. We will give the proof for the second case only as the proof for the first proof is similar. The estimate (2.17) follows directly from Proposition 2.6 and 2.7 for $s' \in (1-s, \frac{n-1}{2})$. To prove the estimate for $s' \in [\frac{n-1}{2}, \frac{n+1}{2})$, we observe that $s' - 2 \in (1-s, \frac{n-1}{2})$ and we have

$$|(I - \Delta_{\partial\Omega})^{-1}((-\Delta_{\partial\Omega})^{\frac{1}{2}} - \mathcal{N})(I - \Delta_{\partial\Omega})|_{L(H^{\frac{n+1}{2}}(\partial\Omega), H^{\frac{n+1}{2}-\alpha+1}(\partial\Omega))} \leq C.$$

Thus (2.17) follows from the commutator estimate (2.16). □

Decomposition of vector fields. We conclude this section by introducing the velocity field decomposition. Given an L^2 vector field $u : \Omega \to \mathbb{R}^n$, it is standard to decompose it into the divergence free part $v \in L^2$ and the gradient part $-\nabla p$ for $p \in H_0^1(\Omega)$. In fact,

$$-\Delta p = \nabla \cdot u \qquad\qquad v = u + \nabla p. \tag{2.18}$$

For any divergence free vector field $v \in L^2(\Omega)$, the normal component on the boundary $v^\perp \triangleq v \cdot N : \partial\Omega \to \mathbb{R}$ in $H^{-\frac{1}{2}}(\partial\Omega)$ is defined as

$$< v^\perp, \psi >= \int_\Omega v \cdot \nabla \psi_{\mathcal{H}} dx$$

for any $\psi \in H^{\frac{1}{2}}(\partial\Omega)$. By interpolation, for any $s_1 \in [0, s - \frac{1}{2}]$,

$$|v^\perp|_{H^{s_1 - \frac{1}{2}}(\partial\Omega)} \leq C|v|_{H^{s_1}(\Omega)}, \tag{2.19}$$

with C uniform in $\Omega \in \Lambda$. This induces a decomposition of v into two divergence free parts, the rotation part v_r and the irrotational (or gradient) part v_{ir}, as follows

$$v_{ir} = \nabla \mathcal{H} \mathcal{N}^{-1} v^\perp, \qquad\qquad v_r = v - v_{ir}. \tag{2.20}$$

It is easy to verify $v_r,\ v_{ir} \in L^2(\Omega)$ and

$$\nabla \cdot v_r = \nabla \cdot v_{ir} = 0, \quad < v_r, v_{ir} >= 0, \quad v_r^\perp = 0.$$

If v is a divergence free velocity field, v_r component is responsible of the internal rotation and v_{ir} of the motion of the domain.

Notation

$\mathrm{tr}(A)$: the trace of an operator.

A^*: the adjoint operator of an operator.

$A_1 \cdot A_2 = \mathrm{tr}(A_1(A_2)^*)$, for two operators.

$B(S,\epsilon) = \cup_{x\in S} B(x,\epsilon)$: an ϵ-neighborhood of a set S.

D and ∂: differentiation with respect to spatial variables.

∇f: the gradient vector of a scalar function f.

∇_X: the directional directive in the direction X.

$\perp$ and $\top$: the normal and the tangential components of the relevant quantities.

$\mathbf{D}_t = \partial_t + v^i \partial_{x^i}$ the material derivative along the particle path.

$\mathbf{D}_t^\top$: the projection of $\mathbf{D}_t$ to the tangent space of $\partial\Omega_t \subset \mathbb{R}^n$.

$N(t,x)$: the outward unit normal vector of $\partial\Omega_t$ at $x \in \partial\Omega_t$.

Π: the second fundamental form of $\partial\Omega_t$, $\Pi(t,x)(w) = \nabla_w N \in T_x\partial\Omega_t$.

$\Pi(X,Y) = \Pi(X)\cdot Y$.

κ: the mean curvature of $\partial\Omega_t$, i.e. $\kappa = \mathrm{tr}\Pi$.

$f_{\mathcal{H}} = \mathcal{H}(f)$: the harmonic extension of f on Ω_t.

$\mathcal{N}(f) = \nabla_N \mathcal{H}(f) : \partial\Omega \to \mathbb{R}$: the Dirichlet-Neumann operator.

$\bar{X} = X \circ u^{-1}$ the Lagrangian coordinates description of X.

$\mathcal{D}$: the covariant differentiation on $\partial\Omega_t \subset \mathbb{R}^n$.

$\mathcal{D}_w = \nabla_w^\top$, for any $x \in \partial\Omega_t \quad w \in T_x\partial\Omega_t$.

$\mathcal{R}(X,Y),\ \ X,Y \in T_x\partial\Omega_t$: the curvature tensor of $\partial\Omega_t$.

$\Delta_{\mathcal{M}} \triangleq \mathrm{tr}\mathcal{D}^2$: the Beltrami-Lapalace operator on a Riemannian manifold $\mathcal{M}$.

Δ^{-1}: the inverse Laplacian with zero Dirichlet data.

$\Gamma = \{\phi : \Omega_t \to \mathbb{R}^n \ ;\ \text{volume preserving homeomorphism}\}$

$\bar{\mathscr{D}}$: the covariant derivative on Γ,

$\mathscr{D}$: represent $\bar{\mathscr{D}}$ in Eulerian coordinates.

$\bar{\mathscr{R}}$: the curvature operator on Γ.

$\mathscr{R}$: represent $\bar{\mathscr{R}}$ in Eulerian coordinates.

II: the second fundamental form of $\Gamma \subset L^2$

$\mathrm{II}_u(w_1, w_2) = \nabla^{\perp}_{w_1} w_2$, for any $u \in \Gamma, \quad w_1, w_2 \in T_u\Gamma$

$p_{v,w} = -\Delta^{-1}\mathrm{tr}(DvDw)$.