Euler's formula

Euler's formula asserts that

$$e^{a+ib} = e^a(\cos(b) + i\sin(b)) \tag{1}$$

where $i^2 = -1$.

The exponential thus defined satisfies many of the usual formulas. Notably

$$\frac{d}{dt}e^{\alpha t} = \alpha e^{\alpha t} \quad e^{\alpha}e^{\beta} = e^{\alpha+\beta}$$

There are many justifications for the formula, which we will not go into.

We just want to show that, this formula, together with the standard formulas for the exponential can be used to unify many other formulas involving trigonometric functions. This unification is particularly useful in differential equations.

1. The most beatiful formula ever

The following formula ties up all the important numbers of analysis:

$$e^{i\pi} + 1 = 0$$

2. Trigonometric formulas

2.1. **Expressions for** sin cos**.** Two particular cases of (**??**) are:

$$e^{ix} = \cos(x) + i\sin(x)$$
$$e^{-ix} = \cos(x) - i\sin(x)$$

Adding and subtracting, we obtain

$$\begin{aligned} \cos(x) &= \frac{1}{2}(e^{ix} + e^{-ix}) \\ \sin(x) &= \frac{1}{2i}(e^{ix} - e^{-ix}) \end{aligned} \tag{2}$$

2.2. **Formulas for addition of angles.** Applying Euler's formula to $e^{i(x+y)} = e^{ix} \cdot e^{iy}$ we obtain

$$\begin{aligned} \cos(x+y) + i\sin(x+y) &= (\cos(x) + i\sin(x)) \cdot (\cos(y) + i\sin(y)) \\ &= \cos(x)\cos(y) - \sin(x)\sin(y) + i(\cos(x)\sin(y) + \sin(x)\cos(y)) \end{aligned}$$

Equating the real and imaginary parts, we obtain the usual formulas for addition of angles.

2.3. **Formulas for powers of trigonometric functions.** Products of trigonometric functions can usually be simplified by using the identities (**??**).

For example, we obtain:

$$\begin{aligned} \cos(x)^2 &= \frac{1}{4}(e^{ix} + e^{-ix})^2 = \frac{1}{4}(e^{2ix} + e^{-2ix} + 2) \\ &= \frac{1}{2}\cos(2x) + \frac{1}{2} \end{aligned} \tag{3}$$

3. Derivatives

From $\frac{d}{dx}e^{ix} = ie^{ix}$, we obtain

$$\begin{aligned}\frac{d}{dx}\cos(x) + i\frac{d}{dx}\sin(x) &= i(\cos(x) + i\sin(x)) \\ &= -\sin(x) + i\cos(x)\end{aligned}$$

4. Integrals

From $\int e^{ix}\,dx = \frac{1}{i}e^{ix} = -ie^{ix}$ we obtain

$$\int \cos(x)\,dx + i\int \sin(x)\,dx = \sin(x) - i\cos(x) + C$$

which are the usual integration formulas.

For integrals involving products of trigonometric functions and exponentials it is quite convenient to use (**??**). Here are some examples of integrals that required different methods.

$$\begin{aligned}\int \cos(x)^2\,dx = &= \int \frac{1}{4}(e^{ix} + e^{-ix})^2\,dx = \int \frac{1}{4}(e^{2ix} + e^{-2ix} + 2)\,dx \\ &= \frac{1}{8i}e^{2ix} - \frac{1}{8i}e^{-2ix} + \frac{1}{2}x + C = \frac{1}{4}\cos(2x) + \frac{1}{2}x + C\end{aligned}$$

$$\begin{aligned}\int \cos(ax)e^{bx}\,dx &= \int \frac{1}{2}(e^{iax} + e^{-iax})e^{bx}\,dx = \frac{1}{2}\int e^{(ai+b)x} + e(-ai+b)x\,dx \\ &= \frac{1}{2}\frac{(-ai+b)e^{(ai+b)x} + (ai+b)e^{(-ai+b)x}}{(-ai+b)(ai+b)} + C \\ &= \frac{1}{2}\frac{e^{bx}(-ai+b)(\cos(ax) + i\sin(ax)) + e^{bx}(ai+b)(\cos(ax) - i\sin(ax))}{(-ai+b)(ai+b)} + C \\ &= \frac{e^{bx}\big(b\cos(ax) + a\sin(ax)\big)}{a^2+b^2} + C\end{aligned}$$

5. Exercises

Compute

(1)

$$\int e^{ax}\sin(bx)\,dx$$

(2)

$$\int \cos(2x)\sin(3x)\,dx$$

(3)

$$\int_{-\pi}^{\pi} \cos(2x)\cos(5x)\,dx$$

(4)

$$\int_{-\pi}^{\pi} \sin(2x)\sin(5x)\,dx$$

Notice that the definite integrals are much easier than the indefinite ones and that can be done without doing the indefinite integrals.

These integrals are very important in Fourier series and in the treatment of power systems, etc.