HOMEWORK 5

2.10.7)

$$\frac{d^2 u}{dx^2} = \frac{1}{k}\frac{du}{dt} \quad 0<x \quad 0<t$$

$$u(0,t) = T_0 \qquad 0<t$$

$$u(x,0) = f(x) \quad 0<x.$$

$$V(x) = Ax + B$$

$$V(0) = B = T_0$$

Steady State should be bounded at $x \to \infty$ $\quad V(x) = T_0$

$$\frac{d^2 w}{dx^2} = \frac{1}{k}\frac{dw}{dt}$$

we know

$$\phi_n(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x.$$

$$w(x,t) = \int_0^\infty B(\lambda) \sin(\lambda x) e^{-\lambda^2 kt} \, d\lambda.$$

$$w(x,0) = f(x) - T_0$$

$$\therefore \; B(\lambda = \frac{2}{\pi} \int_0^\infty f(x - T_0) \sin(\lambda x \, dx.$$

$$U(x,t) = T_0 + \int_0^\infty B(\lambda) \sin(\lambda x) e^{-\lambda^2 kt} \, d\lambda$$

3.1.4)

$$T(x,t)\sin(\phi(x,t)) + T(x+\Delta x,t)\sin(\phi(x+\Delta x,t)) - mg - r\frac{\partial v}{\partial t}$$

$$= m\frac{\partial^2 v}{\partial t^2}(x,t)$$

$$T(x,t) = \frac{T}{\cos\phi(x,t)}, \quad T(x+\Delta x,t) = \frac{T}{\cos\phi(x+\Delta x,t)}$$

$$-T\tan\phi(x,t) + T\tan\phi(x+\Delta x,t)$$
$$-\delta\,\Delta x\,g + \kappa\,\Delta x\,\frac{\partial v}{\partial t} = \partial\,\Delta x\,\frac{\partial^2 v}{\partial t^2}(x,t)$$

as $\Delta x \to 0$

$$\frac{T}{\delta}\frac{\partial^2 v}{\partial x^2} + \frac{\kappa}{\delta}\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial t^2} + g$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 v}{\partial t^2} + g - \frac{k}{s}\frac{\partial v}{\partial t}\right.$$

3.2.5) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$ $\quad 0 < x < a \quad 0 < t$

$u(0,t) = 0 \qquad u(a,t) = 0$

$$u(x,0) = \begin{cases} U_0 & 0 < x < a/2 \\ 0 & a/2 < x < a \end{cases}$$

$\dfrac{\partial u}{\partial t}(x,0) = 0 \qquad 0 < x < a$

$\phi''(x)\, T(t) = \dfrac{1}{c^2}\,\phi(x)\, T''(t) \Rightarrow \phi^2 + \lambda^2 \phi = 0$

$\phi(0) = 0$
$\phi(a) = 0 \Rightarrow \lambda = \dfrac{n\pi}{a}$

$U_n(x,t) = \sum \sin(\lambda_n x)\left[a_n \cos(\lambda_n c t) + b_n \sin(\lambda_n c t)\right]$

$U(x,0) = f(x) \Rightarrow a_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\sin(\lambda_n x)\,dx.$

$= \dfrac{2U_0}{n\pi}\left[1 - \cos\left(\dfrac{n\pi}{2}\right)\right]$

$b_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\sin(\lambda_n x)\cos(\lambda_n c t)}{n\pi}\, 2U_0\left(1 - \cos\left(\frac{n\pi}{2}\right)\right] \text{ for } \lambda_n = \frac{n\pi}{a}$$

3.5.4

$$\begin{cases} \phi'' + \frac{\lambda^2}{x^4}\phi = 0 \\ \phi(1) = 0 \quad \phi(2) = 0 \end{cases}$$

$$\phi(x) = ax\cos\left(\frac{\lambda}{x}\right) + bx\sin\left(\frac{\lambda}{x}\right)$$

$$\phi'(x) = a\cos\left(\frac{\lambda}{x}\right) - \frac{\lambda a x}{x^2}\sin\left(\frac{\lambda}{x}\right) + b\sin\left(\frac{\lambda}{x}\right) + \frac{\lambda b x}{x^2}\cos\left(\frac{\lambda}{x}\right)$$

$$\phi''(x) = \frac{-\lambda^2}{x^3}\left[a\cos\left(\frac{\lambda}{x} + b\sin\frac{\lambda}{x}\right]\right.$$

$$\phi'' + \frac{\lambda^2}{x^4}\phi = 0$$

$\phi(1) = 0 \Rightarrow \tan(\lambda) = -a/b$

$\phi(2) = 0 \Rightarrow \tan(\lambda/2) = -a/b)$

Solving for λ, we get $\lambda_n = 2n\pi$

$$\phi_n(x) = bx\sin\left(\frac{2n\pi}{x}\right)$$

3·4·5

$$\frac{\partial}{\partial x}\left(s(x)\frac{\partial u}{\partial x}\right) = \frac{p(x)}{c^2}\left(\frac{\partial^2 u}{\partial t^2} + r\frac{\partial u}{\partial t}\right) + q(x)\,u.$$

$$\alpha_1 (u(l,t)) - \alpha_2 \frac{\partial u}{\partial x}(\lambda,t) = c_1 \qquad 0 < t$$

$$\beta_1 u(r,t) + \beta_2 \frac{\partial u}{\partial x}(r,t) = c_2$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Let $\quad u(x,t) = v(x) + w(x,t)$

$$\therefore (Sv')' = qv$$

$w(x,t)$ satisfies.

$$\frac{\partial}{\partial x}\left(s(x)\frac{\partial w}{\partial x}\right) = \frac{p(x)}{c^2}\left(\frac{\partial^2 w}{\partial t^2} + \frac{r\,\partial w}{\partial t}\right) + q(x)\,w.$$

$$w(x,0) = f(x) - v(x)$$

$$w'(x,0) = g(x).$$

Homogenous $\quad w(x,t) = \phi(x)\,T(t).$

$$\Rightarrow \frac{(s(x)\,\phi'(x))'}{\phi(x)\,p(x)} = -\frac{q(x)}{p(x)}$$

$$\rightarrow T''(t) + r\,T'(t) + c^2\lambda^2 T = 0$$

from orthogonality

$$\int_l^r \phi_n(x)\,\phi_m(x)\,p(x)\,dx = 0.$$

$$m^2 + rm + c^2\lambda^2 = 0 \quad (\text{Let } T(t) = e^{mt})$$

$$m_{1,2} = \frac{-r \pm \sqrt{r^2 - 4c^2\lambda^2}}{2}$$

$$T(t) = a_1 e^{m_1 t} + b_1 e^{m_2 t}.$$

$$w(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

$$\frac{\partial w}{\partial t}(x,0) = g(x)$$

$$a_n = \frac{1}{\cancel{f_n}} \int_{l}^{r} [f(x) - v(x)] \, \phi_n(x) \, \rho(x) \, dx.$$

$$b_n = \frac{1}{\cancel{f_n} \lambda_n c} \int_{l}^{r} g(x) \, \phi_n(x) \, dx.$$

$$f_n = \int_{l}^{r} \phi_n^2(x) \, \rho(x) \, dx$$

$$w(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \left[a_n e^{m_1 t} + b_n e^{m_2 t} \right]$$