HOMEWORK – #4 SOLUTIONS

2.5.6

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t} + \gamma^2 u & 0 < x < a \\ u(0,t) = 0 \qquad \dfrac{\partial u}{\partial x}(a,t) = 0 \\ u(x,0) = T_0 \end{cases}$$

Steady State Solutions

$v_{xx} = \gamma^2 v$

$v(0) = 0 \ ; \quad v'(a) = 0$

$v(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x} \Rightarrow$ Solution is $v(x) = 0$

$w(x,t) = u(x,t) - v(x)$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t} + \gamma^2 w.$$

$w(0) = 0, \quad \dfrac{\partial w}{\partial t}(a) = 0$

$w(x,0) = T_0$

$\phi''(x)\,T(t) = \frac{1}{k}\phi(x)\,T'(t) + \gamma^2 \phi(x)\,T(t)$

$\phi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$

$\phi(0) = c_1$

$\phi'(a) = c_2 \dfrac{(2n-1)\pi}{2a} \cos\left(\dfrac{(2n-1)\pi}{2a} x\right)$

$\phi_n(x) = \sin\left(\left(\dfrac{2n-1}{2a}\right)\pi x\right)$

$$T_n(t) = e^{-k(r^2 + \lambda_n^2)t}.$$

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2a}\right) e^{-k\left(r^2 + \left(\frac{2n-1}{2a}\right)^2\right)t}$$

$$b_n = \frac{2}{a}\int_0^a T_0 \sin\left(\frac{(2n-1)\pi x}{2a}\right) dx$$

$$= \frac{4T_0}{(2n-1)\pi}.$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4T_0}{(2n-1)\pi} \sin\left(\frac{(2n-1)}{2a}\pi x\right) \cdot e^{-k\left(r^2 + \left(\frac{2n-1}{2a}\right)^2 \pi^2\right)t}$$

2.6.7

$$b_m = \frac{\int_0^a g(x) \sin(\lambda_m x)\, dx}{\int_0^a \sin^2(\lambda_m x)\, dx}$$

$$\int_0^a g(x) \sin(\lambda_m x)\, dx = \left[-\cos(\lambda_m x) \frac{1}{\lambda_m}\right]_0^a$$

$$= \frac{1}{\lambda_m}\left[1 - \cos(\lambda_m a)\right]$$

$$\int_0^a \sin^2(\lambda_m x)\, dx = \frac{1}{2}\left[x - \frac{\sin(2\lambda_m x)}{2\lambda_m}\right]_0^a$$

$$= \frac{a}{2} - \frac{\sin(2\lambda_m a)}{4\lambda_m}$$

$$b_m = \frac{2\left[1 - \cos(a\lambda_m)\right]}{\lambda_m\left[a - \frac{1}{2\lambda_m}\sin(2\lambda_m a)\right]}$$

2.7.1

$$\phi(x) = c_1 \cos(\lambda \ln(x)) + c_2 \sin(\lambda \ln(x)).$$

$$\phi(1) = 0 = c_1$$

$$\phi(2) = 0 \quad c_2 \sin(\lambda \ln(2) = n\pi \rightarrow \lambda_1 = \frac{n\pi}{\ln(2)}$$

$$\phi_n(x) = \sin\Big(\underbrace{\frac{n\pi}{\ln(2)}}_{\lambda_n} \ln(x)\Big)$$

$$\int_1^2 \phi_n(x)\, \phi_m(x)\, 1/x\, dx = \int_1^2 \sin(\lambda_n \ln(x)) \sin(\lambda_m \ln(x)) \frac{1}{x}\, dx.$$

$$= \frac{1}{2}\left[\frac{\sin(\lambda_n - \lambda_m)\ln(2)}{(\lambda_n - \lambda_m)} - \frac{\sin(\lambda_m + \lambda_n)\ln(2)}{(\lambda_n + \lambda_m)}\right]$$

$$= \underline{\underline{0}}$$

2.8.3

$$\begin{cases} (e^x \phi')' + e^x \gamma^2 \phi = 0 & 0 < x < a \\ \phi(0) = 0 \; ; \; \phi(a) = 0 \end{cases}$$

Verify $\phi_n(x) = e^{-x/2} \sin\left(\frac{n\pi x}{a}\right)$ with $\gamma_n^2 = \left(\frac{n\pi}{a}\right)^2 + 1/4$

Boundary Conditions ✓

$\phi(0) = e^0 \sin(0) = 0$

$\phi(a) = e^{-a/2} \sin\left(\frac{n\pi a}{a}\right) = 0$

eqn

$$(e^x \phi')' + e^x \gamma^2 \phi = \left\{ e^x \left[\frac{n\pi}{a} e^{-x/2} \cos\left(\frac{n\pi x}{a}\right) - \frac{1}{2} e^{-x/2} \sin\left(\frac{n\pi x}{a}\right) \right] \right\}'$$

$$+ \gamma^2 e^x e^{-x/2} \sin\left(\frac{n\pi x}{a}\right) = 0$$

$$= -\left(\left(\frac{n\pi}{a}\right)^2 + \frac{1}{4} - \gamma^2\right) e^{x/2} \sin\left(\frac{n\pi x}{a}\right) = 0.$$

Non trivial Solution is

$$\gamma_n^2 = \left(\frac{n\pi}{a}\right)^2 + \frac{1}{4}$$

Let $f(x) = 1$ for $0 < x < a$

$$C_n = \frac{\int_0^a f_n \phi_n(x) \, p(x) \, dx}{\int_0^a \phi_n^2(x) \, p(x) \, dx.}$$

$$= \frac{\int_0^a e^{x/2} \sin\left(\frac{n\pi x}{a}\right) dx}{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx.}$$

$$\int_0^a e^{x/2} \sin\left(\frac{n\pi x}{a}\right) dx = \left[e^{x/2} \left(-\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right) \right) \right]_0^a.$$

$$= \frac{\frac{a}{n\pi} \left[e^{\frac{a}{2}} \left[(-1)^{n+1} \right] + 1 \right]}{1 + \left(\frac{a}{2n\pi}\right)^2}$$

$$\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a}{2}.$$

$$C_n = \frac{2n\pi \left[e^{a/2} (-1)^{n+1} + 1 \right]}{(n\pi)^2 + \frac{a^2}{4}}$$